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A TWO-DIMENSIONAL CONTINUUM THEORY FOR ANGLE-PLY LAMINATES

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Abstract—A two-dimensional continuum theory of microstructure is developed for stress analysis of angle-ply laminates under in-plane loading. An example problem is used to evaluate the results of the theory against a reference solution obtained by the finite element method. The results are in satisfactory agreement; they also show that the in-plane stresses reach somewhat higher peak values than reported in previous literature.

The theory is also presented in a simplified version, which is found to be adequate for predicting interlaminar stresses and in-plane stress resultants, but does not give acceptable results for the variation of in-plane stresses through the thickness of the laminations.

INTRODUCTION

Rapidly increasing utilization of advanced composites in aerospace industry has created an urgent need for reliable design methods for fiber-reinforced laminates. Clearly, such methods can come of age only after suitable techniques of stress analysis have been developed. At the present time the choice is limited to numerical solutions (finite elements of finite differences) of the three-dimensional elasticity problem [1–3], and the classical lamination theory.

Numerical solutions are costly and lack the viability to be serious contenders as practical design tools (it must be emphasized that we are dealing with a complex distribution of stress in three dimensions which results in a very large number of equations). The lamination theory, on the other hand, is too simple to be of much value in failure prediction, because it is invalid in regions where failure usually starts, such as boundaries and other discontinuities of the laminate.

Several attempts have been made to devise methods that lie in-between three-dimensional elasticity solutions and the lamination theory. The first theory in this category was presented by Puppo and Evensen [4]. Their model of the composite leads essentially to lamination theory with interlaminar shear deformation. The resulting equations, however, are incomplete in the sense that they provide no means of controlling the transverse shear tractions at the boundaries of the laminate. The model also assumes the in-plane stresses to be constant through the thickness of each lamination, which is an unacceptably poor approximation of the actual stress distribution. Other publications [5–9], although capable of good results, have a major drawback: each of the methods is tailored for a specific boundary value problem and does not lend itself to generalization.

The present paper introduces a two-dimensional continuum theory of microstructure for angle-ply laminates under in-plane loading. Both generalized plane stress and plane strain versions of the theory are presented. The resulting equations provide complete control over the inplane as well as the transverse shear tractions at the boundaries of the laminate, so that their application is not restricted to certain boundary value problems. Moreover, the theory can be adapted to cross-ply laminates by a straightforward transformation of the governing equations. The stresses derived from the theory are shown to compare favorably with a finite element solution of the classical, three-dimensional equations of elasticity, at least for the example problem used.

The microstructure theory is also derived in a simplified version, but its usefulness appears to be limited to problems where the interlaminar stresses are of main interest, since it yields poor results for in-plane stresses at the interfaces.

KINEMATICS OF DEFORMATION

The scope of the work is limited to angle-ply laminates of balanced construction under the action of in-plane loading. A so-called "basic unit" of a balanced laminate is shown in Fig. 1. It

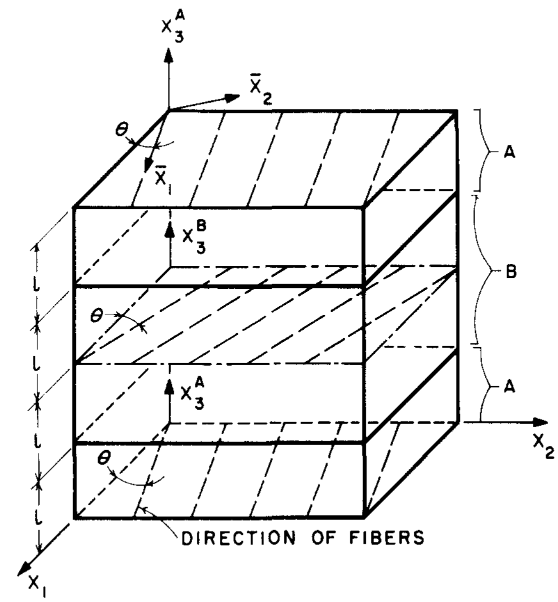

Fig. 1. Basic unit of a balanced angle-ply laminate showing local and global coordinates.

consists of identical, unidirectionally fiber-reinforced layers stacked in a symmetric fashion about the mid-plane of the unit (the layers with ply angles of θ and $-\theta$ are denoted as the A and b-layers, respectively).

A balanced angle-ply laminate is obtained by stacking any number of basic units on top of each other. We postulate that the stress distribution is identical in each basic unit, and independent of the number of units used. Consequently, it is sufficient to study only one basic unit of the laminate. This postulate can be viewed as a generalization of the plane stress, plane strain assumption of the classical elasticity theory.

The technique used for modelling the laminate is the "effective stiffness" approach originally devised by Achenbach and Herrmann[10] for wave propagation, and subsequently refined by Drumheller and Bedford[11]. We start by expanding the displacement field of each layer in terms of polynomials in the transverse (x_3^A or x_3^B) coordinate. It can be shown that the lowest-order, nontrivial model (i.e. a model that does not degenerate into lamination theory) is obtained by truncating the series after the quadratic terms. The displacement components of a typical A or B-layer can thus be written in the form

$$u_i^{A,B}(x_1, x_2, x_3^{A,B}) = \sum_{r=0}^{2} P_r(x_3^{A,B})\Phi_{ir}^{A,B}(x_1, x_2), \; i = 1, 2, 3, \tag{1}$$

where P_r represent the Legendre polynomials

$$P_0(z) = 1, \qquad P_1(z) = z/l, \qquad P_2(z) = (3z^2 - l^2)/(2l^2), \tag{2}$$

and $\Phi_{ir}^{A,B}$ are unknown functions.

Noting that the assumption of generalized plane stress or plane strain implies displacement symmetry about the planes $x_3^A = 0$ and $x_3^B = 0$, we conclude that

$$\Phi_{\alpha 1}^{A,B} = \Phi_{30}^{A,B} = \Phi_{32}^{A,B} = 0, \qquad \alpha = 1, 2. \tag{3}$$

In addition, we should impose continuity of displacements at the interfaces: $u_i^A(x_1, x_2, \pm l) = u_i^B(x_1, x_2, \mp l)$. It is convenient to satisfy continuity of the in-plane displacements by introducing the *generalized displacements* $v_\alpha(x_1, x_2)$, $V_\alpha(x_1, x_2)$, $\chi_\alpha(x_1, x_2)$, $\psi_3(x_1, x_2)$ and $\Psi_3(x_1, x_2)$, such that

$$\Phi_{\alpha 0}^{A,B} = v_\alpha \pm V_\alpha, \qquad \Phi_{\alpha 2}^{A,B} = \chi_\alpha \mp V_\alpha, \qquad \Phi_{31}^{A,B} = \psi_3 \pm \Psi_3, \tag{4}$$

where the upper sign refers to the A-layer and the lower sign to the B-layer. Substituting (3) and (4) in (1), we obtain

$$u_\alpha{}^{A,B} = P_0(v_\alpha \pm V_\alpha) + P_2(\chi_\alpha \mp V_\alpha), \qquad u_3{}^{A,B} = P_1(\psi_3 \pm \Psi_3), \tag{5}$$

which yield the following components of the *strain tensor* for the individual layers:

$$2\epsilon_{\alpha\beta}^{A,B} = P_0[(v_{\alpha,\beta} + v_{\beta,\alpha}) \pm (V_{\alpha,\beta} + V_{\beta,\alpha})] + P_2[(\chi_{\alpha,\beta} + \chi_{\beta,\alpha}) \mp (V_{\alpha,\beta} + V_{\beta,\alpha})], \tag{6a}$$

$$\epsilon_{33}^{A,B} = \frac{1}{l} P_0(\psi_3 \pm \Psi_3), \tag{6b}$$

$$2\epsilon_{\alpha 3}^{A,B} = P_1\left[\left(\frac{3}{l}\chi_\alpha + \underline{\psi_{3,\alpha}}\right) \pm \left(-\frac{3}{l} V_\alpha + \Psi_{3,\alpha}\right)\right]. \tag{6c}$$

Generalized *plane strain* version of the theory is obtained by enforcing continuity also upon u_3, i.e. by setting $\psi_3 = 0$. According to (6b), we then have $\epsilon_{33}^{A} + \epsilon_{33}^{B} = 0$, meaning that the net expansion of the laminate vanishes.

It is now clear that the transverse displacement cannot be made continuous in the case of generalized *plane stress.*

An analogous situation exists in the classical elasticity theory, where all the compatibility equations are imposed upon the plane state of strain, but only in-plane compatibility may be used for plane stress. We do neglect, however, the underlined term in (6c), because otherwise the formulation would not reduce to the classical plane stress model for $\theta = 0$, but takes the form of a strain gradient theory. Again we point out that a similar simplification is used in the classical plane stress theory, where by setting $\epsilon_{13} = \epsilon_{23} = 0$, the gradients of thickness variation are ignored in the computation of stresses.

From here on, all the equations will be derived for the case of plane stress. Modifications of the theory required for plane strain will be presented at the conclusion of the derivations.

EULER EQUATIONS AND BOUNDARY CONDITIONS

The stress-strain relations of the individual layers, referred to the global (x_1, x_2, x_3) coordinate system, are

$$\sigma_{ij}^{A,B} = C_{ijkl}(\pm\theta)\epsilon_{kl}^{A,B} \tag{7}$$

(note that summation convention is used throughout the paper). The elastic stiffness coefficients C_{ijkl} are obtained from the stiffness coefficients $\bar{C}_{ijkl}$ of the local coordinates (principal layer directions) by the usual transformation equations

$$C_{ijkl}(\theta) = Q_{ip}(\theta)Q_{jq}(\theta)Q_{kr}(\theta)Q_{ls}(\theta)\bar{C}_{pqrs}, \tag{8}$$

where

$$\|\underline{Q}(\theta)\| = \begin{Vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

and θ is the ply angle shown in Fig. 1. The explicit expressions for the non-vanishing C_{ijkl} can be found in Ref. [12].

The constants C_{1323} and C_{1233} are usually an order of magnitude smaller than the other stiffness coefficients, enabling us to use the approximation

$$C_{1323} = C_{1233} = 0. \tag{9}$$

It is now a simple matter to compute the average strain energy density E_0 of the laminate:

$$4lE_0 = \int_{-l}^{l} C_{ijkl}(\theta)\epsilon_{ij}^{A}\epsilon_{kl}^{A}\,dx_3{}^A + \int_{-l}^{l} C_{ijkl}(-\theta)\epsilon_{ij}^{B}\epsilon_{kl}^{B}\,dx_3{}^B. \tag{10}$$

In fact, using the orthogonality properties of the Legendre polynomials, the explicit expression for E_0 can be obtained in a single step; the result is given in Appendix A.

The Euler (equilibrium) equations and the natural boundary conditions can now be derived from the *principle of stationary potential energy* by the usual means of variational calculus. The Euler equations take the form

$$t_{\alpha\beta,\alpha}=0, \tag{11a}$$

$$T_{\alpha\beta,\alpha}-S_\beta=0, \tag{11b}$$

$$h_{\alpha\beta,\alpha}-k_\beta=0, \tag{11c}$$

$$L_{\alpha 3,\alpha}-M_3=0, \tag{11d}$$

$$m_3=0, \tag{11e}$$

where comma is used to denote differentiation. The *generalized stresses* appearing in (11a–e) are

$$t_{11}=\frac{\partial E_0}{\partial v_{1,1}}=C_{1111}v_{1,1}+C_{1122}v_{2,2}+C_{1112}(V_{1,2}+V_{2,1})+\frac{1}{l}C_{1133}\psi_3, \tag{12a}$$

$$t_{22}=\frac{\partial E_0}{\partial v_{2,2}}=C_{1122}v_{1,1}+C_{2222}v_{2,2}+C_{2212}(V_{1,2}+V_{2,1})+\frac{1}{l}C_{2233}\psi_3, \tag{12b}$$

$$t_{12}=\frac{\partial E_0}{\partial v_{1,2}}=C_{1212}(v_{1,2}+v_{2,1})+C_{1112}V_{1,1}+C_{2212}V_{2,2}, \tag{12c}$$

$$T_{11}=\frac{\partial E_0}{\partial V_{1,1}}=\frac{6}{5}C_{1111}V_{1,1}+\frac{6}{5}C_{1122}V_{2,2}+C_{1112}(v_{1,2}+v_{2,1})-\frac{1}{5}C_{1112}(\chi_{1,2}+\chi_{2,1})+\frac{1}{l}C_{1133}\Psi_3, \tag{12d}$$

$$T_{22}=\frac{\partial E_0}{\partial V_{2,2}}=\frac{6}{5}C_{1122}V_{1,1}+\frac{6}{5}C_{2222}V_{2,2}+C_{2212}(v_{1,2}+v_{2,1})-\frac{1}{5}C_{2212}(\chi_{1,2}+\chi_{2,1})+\frac{1}{l}C_{2233}\Psi_3, \tag{12e}$$

$$T_{12}=\frac{\partial E_0}{\partial V_{1,2}}=\frac{6}{5}C_{1212}(V_{1,2}+V_{2,1})+C_{1112}v_{1,1}+C_{2212}v_{2,2}-\frac{1}{5}(C_{1112}\chi_{1,1}+C_{2212}\chi_{2,2}) \tag{12f}$$

$$h_{11}=\frac{\partial E_0}{\partial \chi_{1,1}}=\frac{1}{5}[C_{1111}\chi_{1,1}+C_{1122}\chi_{2,2}-C_{1112}(V_{1,2}+V_{2,1})], \tag{12h}$$

$$h_{22}=\frac{\partial E_0}{\partial \chi_{2,2}}=\frac{1}{5}[C_{1122}\chi_{1,1}+C_{2222}\chi_{2,2}-C_{2212}(V_{1,2}+V_{2,1})], \tag{12i}$$

$$h_{12}=\frac{\partial E_0}{\partial \chi_{1,2}}=\frac{1}{5}[C_{1212}(\chi_{1,2}+\chi_{2,1})-C_{1112}V_{1,1}-C_{2212}V_{2,2}], \tag{12j}$$

$$L_{13}=\frac{\partial E_0}{\partial \Psi_{3,1}}=\frac{1}{3}C_{1313}\left(-\frac{3}{l}V_1+\Psi_{3,1}\right), \tag{12k}$$

$$L_{23}=\frac{\partial E_0}{\partial \Psi_{3,2}}=\frac{1}{3}C_{2323}\left(-\frac{3}{l}V_2+\Psi_{3,2}\right), \tag{12l}$$

$$S_1=\frac{\partial E_0}{\partial V_1}=\frac{1}{l}C_{1313}\left(\frac{3}{l}V_1-\Psi_{3,1}\right)=-\frac{3}{l}L_{13}, \tag{12m}$$

$$S_2=\frac{\partial E_0}{\partial V_2}=\frac{1}{l}C_{2323}\left(\frac{3}{l}V_2-\Psi_{3,2}\right)=\frac{3}{l}L_{23}, \tag{12n}$$

$$k_1=\frac{\partial E_0}{\partial \chi_1}=\frac{3}{l^2}C_{1313}\chi_1, \tag{12o}$$

$$k_2=\frac{\partial E_0}{\partial \chi_2}=\frac{3}{l^2}C_{2323}\chi_2, \tag{12p}$$

$$M_3=\frac{\partial E_0}{\partial \Psi_3}=\frac{1}{l^2}C_{3333}\Psi_3+\frac{1}{l}C_{1133}V_{1,1}+\frac{1}{l}C_{2233}V_{2,2}, \tag{12q}$$

$$m_3=\frac{\partial E_0}{\partial \psi_3}=\frac{1}{l^2}C_{3333}\psi_3+\frac{1}{l}C_{1133}v_{1,1}+\frac{1}{l}C_{2233}v_{2,2}. \tag{12r}$$

In (12a–r) we used the notation $C_{ijkl} = C_{ijkl}(+\theta)$.

Solving (11e) for ψ_3 and substituting the result in (12a, b), we obtain

$$t_{11} = \tilde{C}_{1111}v_{1,1} + \tilde{C}_{1122}v_{2,2} + C_{1112}(V_{1,2} + V_{2,1}), \tag{13a}$$

$$t_{22} = \tilde{C}_{1122}v_{1,1} + \tilde{C}_{2222}v_{2,2} + C_{2212}(V_{1,2} + V_{2,1}), \tag{13b}$$

where

$$\tilde{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - \frac{C_{\alpha\beta33}C_{\gamma\delta33}}{C_{3333}} \tag{14}$$

and the usual reduced stiffness coefficients of classical plane stress theory.

The boundary conditions obtainable from the variational problem are

$$t_{\alpha\beta}n_\alpha = t^*_\beta \quad \text{or} \quad v_\beta = v^*_\beta, \tag{15a}$$

$$T_{\alpha\beta}n_\alpha = T^*_\beta \quad \text{or} \quad V_\beta = V^*_\beta, \tag{15b}$$

$$h_{\alpha\beta}n_\alpha = h^*_\beta \quad \text{or} \quad \chi_\beta = \chi^*_\beta, \tag{15c}$$

$$L_{\alpha 3}n_\alpha = L^*_3 \quad \text{or} \quad \Psi_3 = \Psi^*_3, \tag{15d}$$

where $\mathbf{n}$ is the unit outward normal of the boundary, and an asterisk denotes the prescribed boundary tractions and displacements.

If we take $\theta = 0$, then $C_{1112} = C_{2212} = 0$ and $t_{\alpha\beta}$ become identical to the stresses of the classical (orthotropic) plane stress elasticity theory. Moreover, the Euler equations for the "classical" displacements v_α will become uncoupled from the "micro-displacements" V_α, χ_α and Ψ_3.

If $\theta \neq 0$, but the state strain is homogeneous, the Euler equations can be satisfied with $V_\alpha = \chi_\alpha = \Psi_3 = 0$. It can be shown that the in-plane stresses now become $\sigma^{A,B}_{\alpha\beta} = t_{\alpha\beta} \pm T_{\alpha\beta}$ (the remaining stresses are zero), which are identical to the results of the classical lamination theory. We point out, however, that although the lamination theory is capable of satisfying the "classical" boundary conditions (15a), it cannot generally satisfy the stress-boundary conditions (15b) at the same time. Consequently, a free boundary, for example, will give rise to a "boundary-layer" effect, where the generalized displacements V_α, χ_α and Ψ_3 do not vanish.

RECOVERY OF STRESSES

The distribution of in-plane stresses can be obtained by substituting the strains (6a) into the constitutive (7). The result can be arranged in the form

$$\sigma^{A,B}_{\alpha\beta} = P_0(t_{\alpha\beta} \pm T_{\alpha\beta}) + 5P_2h_{\alpha\beta} \pm (P_0 + 5P_2)H_{\alpha\beta}, \tag{16}$$

where

$$H_{11} = \frac{1}{5}[-C_{1111}V_{1,1} - C_{1122}V_{2,2} + C_{1112}(\chi_{1,2} + \chi_{2,1})], \tag{17a}$$

$$H_{22} = \frac{1}{5}[-C_{1122}V_{1,1} - C_{2222}V_{2,2} + C_{2212}(\chi_{1,2} + \chi_{2,1})], \tag{17b}$$

$$H_{12} = \frac{1}{5}[-C_{1212}(V_{1,2} + V_{2,1}) + C_{1112}\chi_{1,1} + C_{2212}\chi_{2,2}], \tag{17c}$$

the remaining variables being the generalized stresses defined previously.

A characteristic drawback of effective stiffness theories is that the stresses are not completely controllable at the boundaries. Equation (16) is no exception due to the presence of the forces $H_{\alpha\beta}$, which did not appear in any of the equilibrium equations or boundary conditions. They are workless forces in the sense that their contribution to the incremental work

$$\delta W = \int_{-l}^{l} \sigma^A_{\alpha\beta}n_\alpha \delta u_\beta{}^A \, \mathrm{d}x_3{}^A + \int_{-l}^{l} \sigma^B_{\alpha\beta}n_\alpha \delta u_\beta{}^B \, \mathrm{d}x_3{}^B$$

vanishes. Consequently, $H_{\alpha\beta}$ may be viewed as forces of internal constraint, their sole function being the enforcement of interlaminar compatibility of the displacements.

Using the same procedure for the interlaminar shear stresses, we get

$$\sigma_{\alpha 3}^{A,B} = P_1(\underline{lk_\alpha} \pm 3L_{\alpha 3}). \tag{18}$$

The underlined term represents stress discontinuity at the interface (another familiar fault of effective stiffness methods), and is therefore neglected.

The procedure cannot be repeated for the interlaminar normal stress for reasons that become apparent upon inspection of $u_3^{A,B}$ in (5). We note that the entire lateral expansion of the laminate is taken up by the displacement discontinuity at the interface, leaving $u_3^A = 0$ on the exterior surfaces $x_3^A = 0$. This means that any lateral pressure applied to the exterior surfaces would be workless and never appear in the potential energy of the laminate. As a consequence, we have no direct control over σ_{33} at $x_3^A = 0$, thus precluding its recovery from the constitutive equations.

The interlaminar normal stress can, however, be obtained by integrating the equilibrium equation $\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0$. Substituting for the shear stresses from (18) and utilizing (11d), the result becomes

$$\sigma_{33}^{A,B} = \left[-\frac{3}{2}P_0 \pm (P_0 - P_2)\right]lM_3. \tag{19}$$

A NOTE ON GENERALIZED PLANE STRAIN

It was shown previously that the generalized plane strain version of the theory is obtained by setting $\psi_3 = 0$ at the outset. The generalized stresses are still given by (12a–r); the Euler eqns (11a–d) also remain valid, but note that now $m_3 \neq 0$, i.e. (11e) is not to be used.

The stress recovery formulas are also unchanged, except that now we must add the plane strain constraint stress lm_3 to the interlaminar normal stress, i.e. (19) must be replaced by

$$\sigma_{33}^{A,B} = \left[-\frac{3}{2}P_0 \pm (P_0 - P_2)\right]lM_3 + P_0 lm_3. \tag{20}$$

EXAMPLE PROBLEM

We illustrate the application of the theory with a simple problem that has been previously studied by several investigators [1, 2, 4, 6, 7, 9]—a semi-infinite laminate under a homogeneous, uniaxial strain, as shown in Fig. 2. The properties of the laminate were chosen as follows: $\theta = 45°$, $\bar{E}_{11} = 13.8 \times 10^{10}$ N/m^2 (20×10^6 psi), $\bar{E}_{22} = \bar{E}_{33} = 1.45 \times 10^{10}$ N/m^2 (2.1×10^6 psi), $\bar{G}_{12} = \bar{G}_{23} = \bar{G}_{13} = 0.586 \times 10^{10}$ N/m^2 (0.85×0^6 psi) and $\bar{\nu}_{12} = \bar{\nu}_{23} = \bar{\nu}_{13} = 0.21$.

A particular integral of the Euler eqns (11a–d) that yields the prescribed strain $\epsilon_{11} = \epsilon_0$ is the solution of the lamination theory

$$v_1^0 = \epsilon_0 x_1, \quad v_2^0 = -(\tilde{C}_{1122}/\tilde{C}_{2222})\epsilon_0 x_2, \quad V_\alpha^0 = \chi_\alpha^0 = \Psi_\alpha^0 = 0, \tag{21}$$

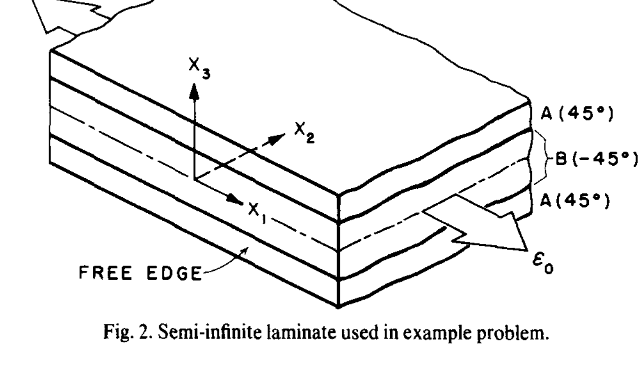

Fig. 2. Semi-infinite laminate used in example problem.

which yields the following non-vanishing generalized stresses:

$$t^0_{11} = (\tilde{C}_{1111} - \tilde{C}^2_{1122}/\tilde{C}_{2222})\epsilon_0, \quad T^0_{12} = (C_{1112} - \tilde{C}_{1122}C_{2212}/\tilde{C}_{2222})\epsilon_0. \tag{22}$$

As T^0_{12} does not vanish at the free boundary, we must superimpose on (21) a solution of the boundary layer type.

For the specific problem at hand, the boundary layer solution takes the one-dimensional form

$$\begin{aligned} v_2 &= A \exp(-px_2/l), & V_1 &= B \exp(-px_2/l), \\ \chi_2 &= C \exp(-px_2/l), & v_1 &= V_2 = \chi_1 = \Psi_3 = 0, \end{aligned} \tag{23}$$

where p must have a positive real part. If (23) are used in (11a–d), three homogeneous equations in A, B And C will be obtained (the remaining Euler equations will be satisfied trivially). Solution of the resulting eigenvalue problem will yield two positive, real values of p and the corresponding eigenvectors (values of A/B and C/B for each value of p). The two values of B are then found from the boundary conditions at $x_2 = 0$:

$$h_{22} = 0, \qquad T_{12} = -T^0_{12}, \tag{24}$$

the remaining boundary conditions being satisfied trivially.

The results of the computation are shown in Figs. 3–6, together with the displacements and stresses obtained from the solution of the classical equations of elasticity by the finite element method (see Appendix B for details). The displacement component u_1 of the finite element solution (Fig. 3) was found to be perfectly skew-symmetric about the interface and symmetric about the center of each layer, thereby confirming the assumptions made about the in-plane displacements. There is, however, a considerable departure in u_3 from the assumed linear distribution near the free edge (Fig. 4).

The in-plane stresses of the finite element solution have extremely steep gradients along the interface (Fig. 5), culminating in peak stresses just inside the free boundary. These peak values appear to be considerably greater than reported previously[1]. The corresponding stresses obtained from the microstructure theory are not quite able to cope with such high gradients, but still manage to show creditable correlation. Note that σ_{12} does not vanish at the free boundary for reasons explained previously.

Figure 5 also compares the average in-plane stresses, computed from

$$\bar{\sigma}^{A,B}_{\alpha\beta} = \frac{1}{2l}\int_{-l}^{l} \sigma^{A,B}_{\alpha\beta}\,dx_3^{A,B} \quad \text{and} \quad \bar{\sigma}^{A,B}_{\alpha\beta} = t_{\alpha\beta} \pm T_{\alpha\beta},$$

for the finite element solution and microstructure theory, respectively.

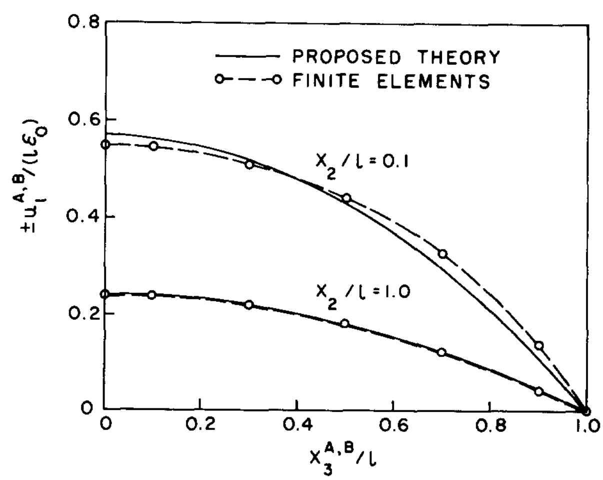

Fig. 3. Distribution of in-plane displacement u_1 through thickness.

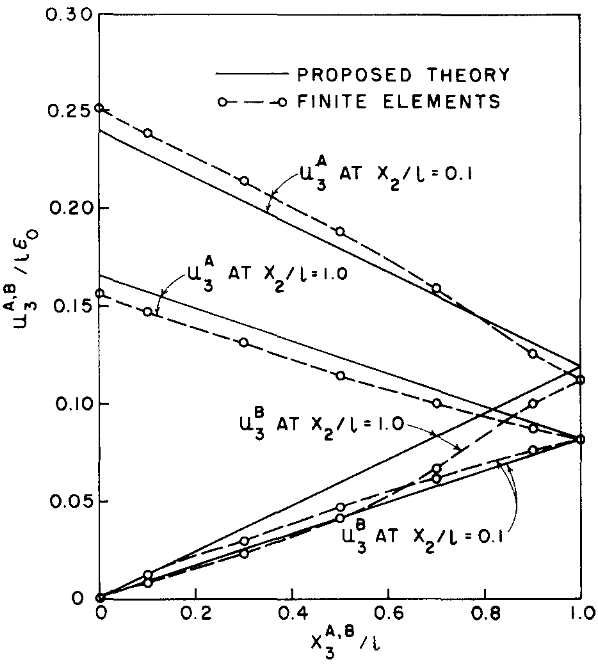

Fig. 4. Distribution of transverse displacement through thickness.

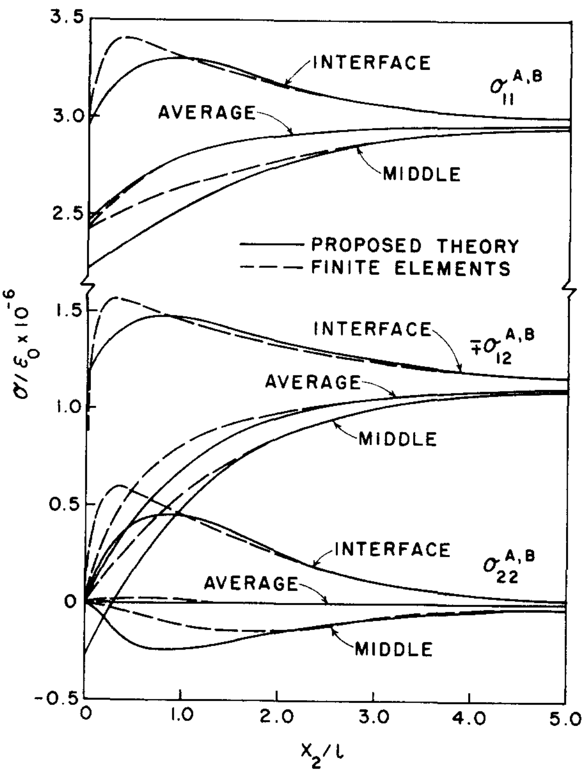

Fig. 5. Distribution of in-plane stresses with distance from free edge.

The interlaminar stresses σ_{13} and σ_{33} (the proposed theory predicts $\sigma_{33} = 0$ everywhere), shown in Fig. 6, are also in satisfactory agreement, except at the interface near the free boundary, where the reference solution predicts very high values (an exact solution of elasticity theory would probably predict a stress singularity). It must be kept in mind, however, that the locally high stresses are largely a result of our assumption of material homogeneity within each layer—see, for example, [13] for a discussion of the topic—and are, therefore, not to be taken literally. The shear stress component σ_{23} has been omitted from the figures because it vanishes in the reference solution as well as in the microstructure theory.

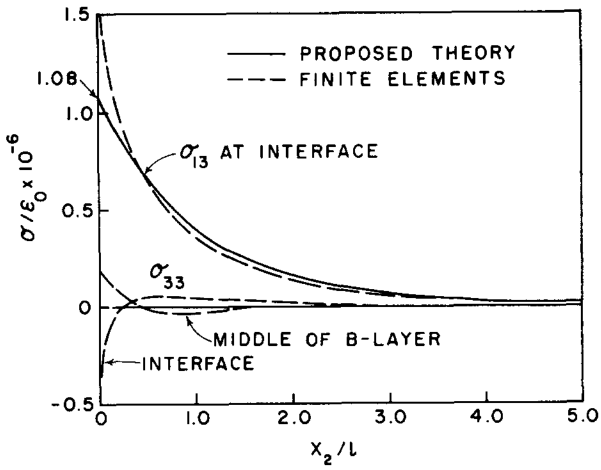

Fig. 6. Distribution of interlaminar stresses with distance from free edge.

SIMPLIFIED VERSION OF THE THEORY

In the preceding theory the interlaminar shear stresses (18) were found to be discontinuous at the interfaces. We will now explore the possibility of removing these discontinuities by imposing appropriate constraints on the generalized displacements. It must be realized, however, that stress continuity is not an intrinsic requirement of the principle of stationary potential energy. The additional constraints are thus to be viewed as simplifying approximations which in general lead to less accurate theory.

According to (18) the continuity condition on the interlaminar shear stresses is $k_\alpha = 0$, which in view of (12o, p) requires that $\chi_\alpha = 0$. If this constraint is imposed at the outset, i.e. in eqn (5), χ_α will never appear in the strain energy of the laminate. Consequently $h_{\alpha\beta}$ and k_α vanish altogether from the Euler equations and boundary conditions.

In summary, the simplified version of the theory is obtained by simply setting

$$\chi_\alpha = h_{\alpha\beta} = k_\alpha = 0 \tag{25}$$

in all the equations derived previously.

The solution of the boundary value problem shown in Fig. 2 is now greatly simplified. The

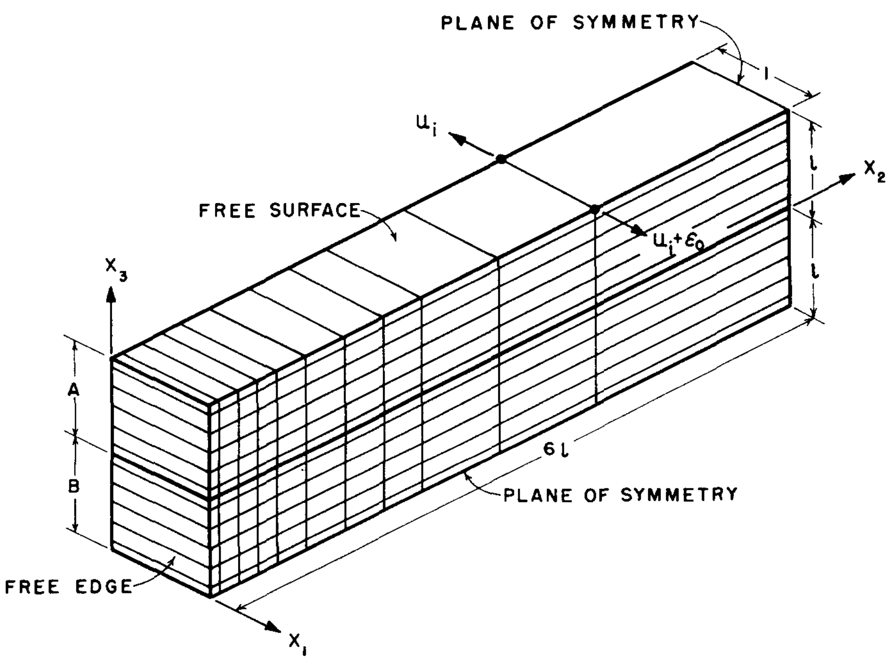

Fig. 7. Finite element model used in reference solution.

Euler eqns (11a, b, d) yield the characteristic solution of the boundary layer problem $p^2 = 3C_{1313}/D$, $A/B = -C_{2212}/\tilde{C}_{2222}$, where $D = (6/5)C_{1212} - C_{2212}^2/\tilde{C}_{2222}$. The boundary condition $T_{12} = -T_{12}^0$ at $x_2 = 0$ then gives $B = T_{12}^0/\sqrt{(3C_{1313}D)}$. The expression for the maximum interlaminar shear is particularly simple: $(\sigma_{13})_{max} = 3C_{1313}B = pT_{12}^0$, occurring at the intersection of the free boundary and the interface.

Detailed results of the simplified theory for the $\pm 45°$ laminate can be found in Ref. [12]. It was shown that σ_{13} is not significantly affected by the simplification. The average in-plane stresses are also virtually unchanged, but the thickness-distribution is grossly in error. The simplification is thus useful only in cases where the interlaminar stresses (or the average in-plane stresses) are of interest.

An interesting observation is that if we also set $\Psi_3 = 0$, the theory can be shown to degenerate into Puppo and Evensen's model of the laminate.

CLOSURE

The theory presented here is the simplest possible continuum representation of an angle-ply laminate in the sense that any further simplification would lead to an unacceptable degeneration.

The main deficiency of the theory is the incompatibility of the transverse displacement at the interlaminar surfaces in the case of plane stress. This discontinuity can be eliminated only by abandoning the generalized plane stress assumption and admitting bending deformations of the individual layers. The resulting theory would not only be quite complex, but also dependent on the number of layers used in its construction.

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APPENDIX A

The average (thickness-average) strain energy density E_0 of the laminate is:

$$
\begin{aligned}
2E_0 = {} & C_{1111}\left(v_{1,1}^2 + \frac{6}{5}V_{1,1}^2 + \frac{1}{5}\chi_{1,1}^2\right) + C_{2222}\left(v_{2,2}^2 + \frac{6}{5}V_{2,2}^2 + \frac{1}{5}\chi_{2,2}^2\right) \\
& + 2C_{1122}\left(v_{1,1}v_{2,2} + \frac{6}{5}V_{1,1}V_{2,2} + \frac{1}{5}\chi_{1,1}\chi_{2,2}\right) \\
& + C_{1212}\left[(v_{1,2}+v_{2,1})^2 + \frac{6}{5}(V_{1,2}+V_{2,1})^2 + \frac{1}{5}(\chi_{1,2}+\chi_{2,1})^2\right] \\
& + 2C_{1112}\left\{(V_{1,2}+V_{2,1})\left(v_{1,1} - \frac{1}{5}\chi_{1,1}\right) + V_{1,1}\left[(v_{1,2}+v_{2,1}) - \frac{1}{5}(\chi_{1,2}+\chi_{2,1})\right]\right\} \\
& + 2C_{2212}\left\{(V_{1,2}+V_{2,1})\left(v_{2,2} - \frac{1}{5}\chi_{2,2}\right) + V_{2,2}\left[(v_{1,2}+v_{2,1}) - \frac{1}{5}(\chi_{1,2}+\chi_{2,1})\right]\right\}
\end{aligned}
$$

$$+\frac{1}{l^2}C_{3333}(\psi_3{}^2+\Psi_3{}^2)$$

$$+\frac{2}{l}C_{1133}(v_{1,1}\psi_3+V_{1,1}\Psi_3)+\frac{2}{l}C_{2233}(v_{2,2}\psi_3+V_{2,2}\Psi_3)$$

$$+\frac{1}{3}C_{1313}\left[\left(\frac{3}{l}\chi_1\right)^2+\left(\frac{3}{l}V_1-\Psi_{3,1}\right)^2\right]$$

$$+\frac{1}{3}C_{2323}\left[\left(\frac{3}{l}\chi_2\right)^2+\left(\frac{3}{l}V_2-\Psi_{3,2}\right)^2\right].$$

APPENDIX B

The finite element reference solution was obtained by the use of the Solid SAP[14] computer program using isoparametric, hexahedral elements. It was necessary to make two modifications to the program:

(i) Orthotropic material properties and the corresponding transformation equations for the elastic stiffness coefficients were added (the original elements were limited to isotropic materials).

(ii) Provision was made for kinematical constraints of the type $u_j = u_i + \delta$, where u_i and u_j are the displacements of specified nodes, and δ is a prescribed value.

The finite element model of the laminate is shown in Fig. 7. The special kinematical constraints enabled us to impose the uniform strain ϵ_0, and still permit warping of the "cross sections" (x_1 = constant planes) of the laminate. Symmetry conditions imposed at $x_2 = 6l$ imply that the results are strictly valid for a laminate specimen of width $12l$. However, the boundary layer is sufficiently narrow so that the effects of a free edge are negligibly small at the center of the strip; i.e. the finite element model will behave essentially as a semi-infinite body.

In order to check the convergence of the finite element solution, the problem was also solved with a finer grid (using twice as many elements as shown in Fig. 7, and with a reversed numbering of the nodes. No significant change in the results was observed.